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ORIGINAL PAPER

On the roots of the trinomial equation

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Abstract Here we summarize the works of the Hungarian mathematician Jenő Egerváry (1891–1958) on the trinomial equations. We present some of his ideas and methods with examples. Some earlier results in the history of mathematics in Hungary about the trinomial equations are also discussed.

Keywords Bolyai algorithm · Egerváry · Equilibrium positions · Trinomial equaitons

1 Introduction

The works of Jenő Egerváry (1891–1958) spanned several fields of mathematics—differential and integral equations, differential geometry, trigonometric polynomials and series, matrix theory, graph theory—, and have various applications. He also wrote a number of articles on algebra Egerváry (1922a,b, 1930, 1931). In the latter he found several important results on the arrangements of the roots of trinomial equations. These publications met with a warm response in the literature Rózsa (1983).

In 1930, Egerváry published a paper Egerváry (1930) on the arrangements of the roots of trinomial equations in the complex plane. He studied the distribution of the roots based on their arguments and moduli. The central idea here came from an interesting observation: *the roots of the trinomial equations can be interpreted as the equilibrium points of unit masses that are located at the vertices of two regular concentric polygons centered at the origin in the complex plane*. Using the symmetry and continuity properties of this force field he showed how the roots could be separated

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according to their arguments. In this separation Egerváry determined sectors in the complex plane, where each sector contains a root of the equation, and the sum of the angles of these sectors is of order $\pi/2$. He also gave a characterization of the distribution of the roots according to their moduli. Egerváry localized the roots in annulus form. The crowning achievement of his work was a synthesis of these results. Based on previous studies, he was able to separate the roots into sectors of annuli.

2 The Bolyai algorithm

The roots of trinomial equations had already been studied in the 19th century in Hungary and elsewhere. In 1786, the Swedish mathematician Erland Samuel Bring (1736–1798) reduced the general quintic equation to a trinomial equation using the Tschirnhaus transformation (Garding 1998; Szénássy 1975, 1992). This result which is used in the calculation of annuities, which grew in importance with the rise of capitalism, motivated several mathematicians and they began to search for the solutions of trinomial equations. After the failure of attempts at solving the general trinomial equation by algebraic means, attention turned to approximation methods. The Hungarian mathematician Farkas Bolyai (1775–1856) proposed an iterative method for the root approximation of a special trinomial equation (Bolyai 1832–1833).

Suppose we have the following equation ($m \in \mathbb{N}, m > 1, a > 0$):

$$x^m = a + x.$$

Now consider the following iteration procedure with an initial value of zero:

$$\begin{aligned} x_0 &= 0 \\ x_1 &= \sqrt[m]{a} \\ x_2 &= \sqrt[m]{a + \sqrt[m]{a}} \\ &\vdots \\ x_{n+1} &= \sqrt[m]{a + x_n} \end{aligned} \tag{1}$$

Consistently taking the positive real value of the root, Bolyai demonstrated that the sequence $\{x_n\}_{n=1}^{\infty}$ is monotonically increasing, it is bounded from above, and that $\lim x_n$ is a solution of the given trinomial equation.

In an early article Farkas (1881) of the well-known Hungarian mathematician and physicist, Gyula Farkas (1847–1930) extended Bolyai's method to a more general trinomial equation ($x^m = a + bx$), taking special care with questions of convergence. Farkas called this iterative method that Bolyai used the *Bolyai algorithm*. He proved the following theorems:

- For $a, b > 0$, the method converges to a root of the equation.
- For $a > 0, b < 0$, and $a > |b\sqrt[m]{a}|$ the algorithm converges to a root if $mx_{2k}^{m-1} > -b$ for each $k \in \mathbb{N}$.
- For $a > 0$, $b < 0$, odd m, and $a < |b\sqrt[m]{a}|$, the iteration sequence is divergent.

The Bolyai algorithm became well known and it inspired several mathematicians to seek its generalization Farkas (1884) and look for applications (e.g. the Bolyai algorithm and the problem of calculating interest rate for annuities). The Bolyai algorithm is very simple, but from a practical point of view it is not the best because its speed of convergence is rather slow. There are more efficient ways available for finding the roots of this equation (Veress 1943).

3 Some earlier results of Egerváry's work

The approximation of the roots was just one direction in the investigation of the trinomial equations. At the end of the nineteenth century a number of mathematicians had already written articles about the arrangements of the roots of trinomial equations in the complex plane. For example Nekrasov (1887) determined *sectors* in the complex plane, where each sector contained a root of the equation. He characterized a distribution of the roots according to their arguments. Bohl (1914) found a method for calculating the number of roots of a trinomial equation in a given *circle* in the complex plane. Bohl's approach is connected with the separation of the roots based on their moduli. Herglotz (1922) studied the *Riemann surfaces* that correspond to the trinomial equations. He found a functional connection between the roots and the coefficients of the equation. Egerváry was quite familiar with these results. In the following sections we will present some results of Jenő Egerváry on the arrangements of the roots of trinomial equations Egerváry (1930). We shall demonstrate his approaches by providing examples.

4 Equivalent trinomial equations

First of all, let us consider the definition of two equivalent trinomial equations. Two trinomial equations are considered equivalent if the systems of their roots differ only in a rotation or reflection.

Definition 1 Two trinomial equations

$$\begin{aligned} f_1(z) &= A_1 z^{n+m} + B_1 z^m + C_1 = 0 \\ f_2(z) &= A_2 z^{n+m} + B_2 z^m + C_2 = 0 \end{aligned} \tag{2}$$

are equivalent if

$$f_1(z) = c f_2(e^{i\delta} z) \text{ or } f_1(z) = c \overline{f}_2(e^{i\delta} z),$$

where $A_{\{1,2\}} \neq 0$, $B_{\{1,2\}} \neq 0$, $C_{\{1,2\}} \neq 0$, $z \in \mathbb{C}$, $c, \delta \in \mathbb{R}$, n and m are relative primes.

Egerváry proved a theorem which determined the relation between the arguments and moduli of coefficients of two equivalent trinomial equations.

Theorem 1 *(Egerváry 1930) Two trinomial equations (2) are equivalent if and only if*

$$\left|\frac{A_1}{A_2}\right| = \left|\frac{B_1}{B_2}\right| = \left|\frac{C_1}{C_2}\right|$$

and

$$m(\alpha_1 \pm \alpha_2) - (n+m)(\beta_1 \pm \beta_2) + n(\gamma_1 \pm \gamma_2) \equiv 0 \pmod{2\pi},$$

where $\alpha_{\{1,2\}}, \beta_{\{1,2\}}, \gamma_{\{1,2\}}$ *are the arguments of* $A_{\{1,2\}}, B_{\{1,2\}}, C_{\{1,2\}}$.

Egerváry found some other important theorems about the class of equivalent trinomial equations. Let us consider two trinomial equations where the moduli are equal in their related coefficients:

$$\begin{aligned} |A|e^{i\alpha_1} z^{n+m} + |B|e^{i\beta_1} z^m + |C|e^{i\gamma_1} = 0 \\ |A|e^{i\alpha_2} z^{n+m} + |B|e^{i\beta_2} z^m + |C|e^{i\gamma_2} = 0 \end{aligned} \tag{3}$$

The Eq. (3) have roots with the same moduli if and only if the two trinomial equations are equivalent. A trinomial equation has roots with the same moduli if and only if the equation is equivalent to a trinomial equation whose coefficients are real numbers. A trinomial equation has a root with higher multiplicity if and only if it is equivalent to a trinomial equation with real coefficients and its coefficients satisfy the following relation:

$$(-1)^{n+m} \frac{A^m C^n}{B^{n+m}} = \frac{m^m n^n}{(m+n)^{m+n}}.$$

5 On the roots of binomial and trinomial equations

It is well known that the roots of the binomial equation

$$z^n + v = 0 \tag{4}$$

lie at the vertices of a regular polygon centered at the origin in the complex plane. The roots of (4) are simply

$$v_j = |v|^{1/n} e^{\frac{\arg v + (2j+1)\pi}{n}} \quad (1 \le j \le n).$$

Let us put unit masses at v_i $(1 \le i \le n)$ in such a way that their force is inversely proportional to their distance. Then the resultant of the forces in z is

$$\sum_{i=1}^{n} \frac{1}{\overline{z} - \overline{v}_i} = \frac{n\overline{z}^{n-1}}{\overline{z}^n + \overline{v}}.$$

In a similar way for the equation $z^{n+m} + w = 0$ this force is

$$\frac{(n+m)\overline{z}^{n+m-1}}{\overline{z}^{n+m} + \overline{w}}.$$

Hence the equilibrium positions of the force field of the unit masses located at the vertices of two concentric regular polygons (centered at the origin) correspond to the roots of the equation

$$\frac{nz^{n-1}}{z^n + v} + \frac{(n+m)z^{n+m-1}}{z^{n+m} + w} = 0.$$

This constraint is equivalent to the following constraint (except for the trivial equilibrium point $z = 0$):

$$(2n+m)z^{n+m} + (n+m)vz^m + nw = 0. \tag{5}$$

An important observation, and it is easy to check, is that each trinomial equation

$$Az^{n+m} + Bz^m + C = 0$$

can be transformed to the following form related to Eq. (5):

$$\frac{A}{2n+m}\left[(2n+m)z^{n+m} + (n+m)\frac{(2n+m)B}{(n+m)A}z^m + n\frac{(2n+m)C}{nA}\right] = 0$$

The following theorem establishes a connection between the roots and the equilibrium points of the force field for the trinomial equation.

Theorem 2 (Egerváry 1930) *The trinomial equation*

$$Az^{n+m} + Bz^m + C = 0$$

with

$$A = |A|e^{i\alpha}, \quad B = |B|e^{i\beta}, \quad C = |C|e^{i\gamma}$$

determines two concentric regular polygons P_{n+m} and P_n in the complex plane. The vertices of P_n are

$$\left(\frac{2n+m}{n+m}\left|\frac{B}{A}\right|\right)^{\frac{1}{n}} e^{\frac{\beta-\alpha+(2\nu+1)\pi}{n}i} \qquad \nu = 1, \ldots, n.$$

The vertices of P_{n+m} are

$$\left(\frac{2n+m}{n}\left|\frac{C}{A}\right|\right)^{\frac{1}{n+m}} e^{\frac{\gamma-\alpha+(2\lambda+1)\pi}{n+m}i} \qquad \lambda = 1, \ldots, n+m.$$

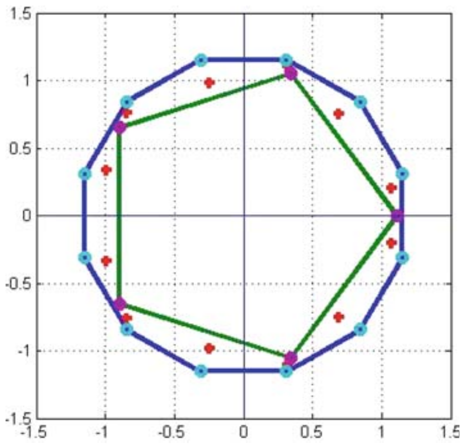

Fig. 1 The roots of the equation $5z^{12} - 6z^7 + 12 = 0$ are located at equilibrium points

The equilibrium points of the force field of the unit masses at the vertices of P_n and P_{n+m} are the roots of the given trinomial equation.

In Fig. 1 one can see the roots (marked by a +) of the equation $5z^{12} - 6z^7 + 12 = 0$, and the polygons P_5 and P_{12}.

6 The distribution of the roots of trinomial equations according to the arguments and moduli

As we mentioned previously, Nekrasov (1887) had already characterized the distribution of the roots of trinomial equations according to their arguments. Nekrasov determined the corresponding sectors in the complex plane, where each sector contains exactly one root of the equation. Egerváry investigated the problem in a similar way, but he used an improved sector decomposition with a smaller sum of angles.

Let us consider $2(n+m)$ half-lines from the origin, each of which have the following arguments:

$$\begin{aligned} \Theta &= \Theta_\lambda^{(n+m)} \equiv \frac{\gamma - \alpha + (2\lambda + 1)\pi}{n + m} \pmod{2\pi}, \\ \Theta &= \Theta_\mu^{(m)} \equiv \frac{\beta - \alpha + (2\mu + 1)\pi}{m} \pmod{2\pi}, \\ \Theta &= \Theta_\nu^{(n)} \equiv \frac{\gamma - \beta + (2\nu + 1)\pi}{n} \pmod{2\pi}, \end{aligned} \tag{6}$$

where $1 \le \lambda \le n + m$, $1 \le \mu \le m$, $1 \le \nu \le n$.
The arguments of (6) determine exactly $n + m$ sectors in the complex plane with the angles:

$$|\Theta_\lambda^{(n+m)} - \Theta_\mu^{(m)}| \le \frac{\pi}{n + m}$$

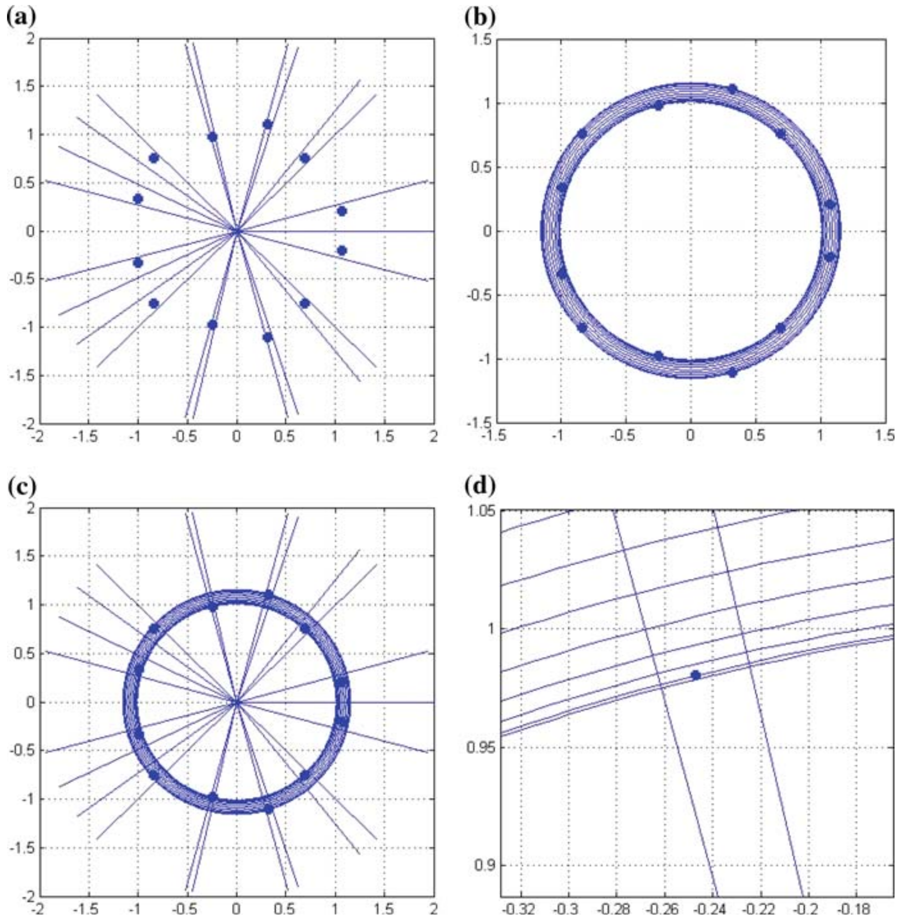

Fig. 2 The roots of $5z^{12} - 6z^7 + 12 = 0$ in sectors (**a**) and annuli (**b**), and sectors of annuli (**c,d**)

and

$$|\Theta_\lambda^{(n+m)} - \Theta_\nu^{(n)}| \le \frac{\pi}{n+m}.$$

Each sector contains exactly one root of the trinomial equation. For the equation $5z^{12} - 6z^7 + 12 = 0$ the corresponding sectors and the roots are shown in Fig 2a.

Egerváry localized in annulus form the roots of trinomial equations based on their moduli. To see how he did it, let us consider those circles centered at the origin where the roots of the following equations

$$\begin{aligned} &|A|z^{n+m} - |B|z^m + |C| = 0 \\ &|A|z^{n+m} + |B|z^m + (-1)^m|C| = 0 \end{aligned} \tag{7}$$

lie on their circumferences. We get $n + m + 1$ circles if

$$\left|\frac{B^{n+m}}{A^m C^n}\right| \leq \frac{(n+m)^{n+m}}{n^n m^m},$$

otherwise we will have $n+m+2$ circles. These circles determine $n+m$ annuli, where each annulus contains exactly one root of the equation (see Fig. 2b).

We can combine the above two approaches. In this case the roots lie in sectors of an annulus (see Fig. 2c, d).

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